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Sufficient Optimality Criterion for Linearly Constrained, Separable Concave Minimization Problems[1]

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Abstract. A sufficient optimality criterion for linearly-constrained concave minimization problems is given in this paper. Our optimality criterion is based on the sensitivity analysis of the relaxed linear programming problem. The main result is similar to that of Phillips and Rosen (Ref. 1); however, our proofs are simpler and constructive.

In the Phillips and Rosen paper (Ref. 1), they derived a sufficient optimality criterion for a slightly different linearly-constrained concave minimization problem using exponentially many linear programming problems. We introduce special test points and, using these for several cases, we are able to show optimality of the current basic solution.

The sufficient optimality criterion described in this paper can be used as a stopping criterion for branch-and-bound algorithms developed for linearly-constrained concave minimization problems.

Key Words. Separable concave minimization problems, linear relaxation, sensitivity analysis.

1. Introduction

We consider separable concave minimization problem in the following form:

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$$\text{(P)}\quad \min \sum_{j=1}^{n} f_j(x_j),$$
$$\text{s.t.}\ Ax \le b,$$
$$l \le x \le u,$$

where $A \in \mathbb{R}^{m\times n}$ is a matrix, $b \in \mathbb{R}^m$, $l, u \in \mathbb{R}^n$ are given vectors, $l \ge 0$, $f_j : \mathbb{R} \to \mathbb{R}$ are concave functions, and $x \in \mathbb{R}^n$ is a vector of unknowns. Let us introduce the sets

$$\mathcal{A} := \{x \in \mathbb{R}^n : Ax \le b\}, \quad \mathcal{T} := \{x \in \mathbb{R}^n : l \le x \le u\}.$$

Then, the set of feasible solutions of problem (P) is defined as

$$\mathcal{P} = \mathcal{A} \cap \mathcal{T},$$

which assumes that the domain of $f_j\, [l_j, u_j] \subseteq \mathcal{D} f_j$ holds. Furthermore, $\mathcal{P}^*$ denotes the set of optimal solution of problem (P). If $\mathcal{P} \neq \emptyset$, then $\mathcal{P}^* \neq \emptyset$ holds, since $\mathcal{P}$ is bounded and closed. Problem (P) is one of the simplest optimization problems that does not belong to the class of convex optimization. This problem has two important theoretical properties: there is an optimal solution at a vertex of the polyhedron $\mathcal{P}$ (Ref. 2); moreover, if f_j is strictly concave, then each optimal solution is a vertex of the polyhedron. Hence, problem (P) is in the class of NP-complete problems (Ref. 3).

Several practical applications can be formulated via problem (P), for instance; certain control problems (Ref. 4), concave knapsack problems (Ref. 5), various production and transportation problems (Ref. 6), production planning problems (Ref. 7), process network synthesis problems (Ref. 8), some network flow problems (Ref. 9).

Due to the importance and applicability of model (P), there is a considerable literature on possible solution methods. In the literature, there are three main types of algorithm: listing the vertices of the polyhedron $\mathcal{P}$, cutting-plane methods, and branch-and-bound algorithms (BB). Several versions of BB are discussed in Refs. 1 and 10–16; vertex enumeration procedures are used in Refs. 17–19 to solve problem (P); cutting-plane algorithms are described in Refs 20–22. There are a number of alternative methods such as approximation using splines (Ref. 23) or combination of BB and cutting-plane algorithms (Ref. 21).

In this paper, a sufficient optimality criterion is given for linearly-constrained separable concave minimization problem. The optimality criterion is based on the linear programming relaxation of (P) and the sensitivity analysis of that linear programming problem. The result obtained is similar to that of Phillips and Rosen (Ref. 1), but our proof is elementary and constructive. Furthermore, it dose not require finding a common optimal solution of exponentially many linear programming problems. We introduce a test which can be used efficiently as stopping criterion or branching criterion in a BB algorithm. The numerical implementation and testing of our BB algorithm is in progress.

Section 2 deals with the linear programming relaxation of problem (P) and the related optimality criteria known from the linear programming literature. In Section 3, using the sensitivity analysis of the linear programming problems, we introduce the sufficient optimality criterion for problem (P). A test point, which may violate these optimality criteria, derived from the sensitivity analysis of linear programming relaxation, is introduced in Section 4. Here 4, we show that the nonexistence of a violating test point for the optimal solution of the relaxed linear programming problem means that the given vertex is the optimal solution of the original problem (P).

In this paper, small (indexed) Latin, sometimes Greek, letters like $x_i, y_i, \gamma, \beta, \ldots$ denote real numbers. Exceptions are $f, f_j, \bar{f}, f'_k, g, g_j$, which are used to denote functions. Also, m is the number of constraints in (P) and n denotes the number of variables used in the problem. Capital Latin letters likes $A, B, \ldots$ denote matrices, while calligraphic letters $\mathcal{A}, \mathcal{P}, \ldots$ denote sets. The n-dimensional Euclidean space is denote by $\mathbb{R}^n$ matrices of size $m \times n$ are associated with the space $\mathbb{R}^{m\times n}$. Elements of sets in $\mathbb{R}^n$ or $\mathbb{R}^m$ are the unknowns of a system of linear inequalities. Bounds and right-hand sides of these inequalities and columns or rows of matrices are all vectors, denoted by lower case letters $x, b, l, u, 0, a_j$ etc. Furthermore, we denote the nonnegative orthant of $\mathbb{R}^n$ by $\mathbb{R}^n_\oplus$; i.e.,

$$\mathbb{R}^n_\oplus = \{x \in \mathbb{R}^n : x \geq 0\}.$$

The positive vectors of the Euclidean space are denoted by $\mathbb{R}^n_+$ and the indices of the unknowns by

$$\mathcal{J} := \{1, \ldots, n\}.$$

Finally, the vector of all ones is represented by the $\vec{e}$.

2. Relaxed Linear Programming Problem

During the solution of problem (P), for instance, using a branch-and-bound algorithm, the first step is to form and solve a linear programming relaxation. By solving the linear programming relaxation and applying the post-optimality analysis to the optimal solution, we can introduce a sufficient optimality criterion of the original problem. In this way, we can decide whether or not optimal solution of the relaxed problem is in fact the optimal solution of the original problem (P).

It is necessary to introduce the following statements about the properties of a one-dimensional concave function (Ref. 24, pages 228).

Proposition 2.1. Let f be a one-dimensional function on the interval $I \subset D_f$. The following statements are equivalent:

(a) f is concave on the interval I.

(b) Let $x,\ y \in I,\ x \neq y$, and

$$m(x, y) = [f(y) - f(x)]/(y - x).$$

If $a,\ b,\ c \in I,\ a < b < c$, then the following holds:

$$m(a, b) \geq m(a, c) \geq m(b, c).$$

(c) For any $t \in I$, the function $m_t(x) = m(t, x)$ is decreasing on $I \backslash \{t\}$.

(d) If $a,\ b,\ c \in I,\ a < b < c$, then

$$m(a, b) \geq m(b, c).$$

The following proposition is an important consequence of the properties listed above (Ref. 24, page 232).

Proposition 2.2. Let f be a one-dimensional concave function on the open interval $I \subset D_f$. Then:

(a) f is continuous on the interval I.

(b) At any $t \in I$, the function is left and right differentiable and

$$f'_-(t) \geq f'_+(t).$$

(c) If $a, b \in I,\ a < b$, then

$$f'_+(a) \geq m(a, b) \geq f'_-(b);$$

moreover, if f is strictly concave on the interval I, then

$$f'_+(a) > m(a, b) > f'_-(b).$$

Sections 2.1 and 2.2 summarize the relaxed linear programming problem generation pertaining to problem (P) in order to calculate a lower bound for problem (P). Although these calculations are well known in the literature, they were included here for the sake of completeness. In Section 2.3, the optimality criteria of the relaxed linear programming problem are obtained.

2.1. Relaxation of the Concave Functions f_j on the Set $\mathcal{T}$. Let us consider the linear relaxation of the concave functions $f_j : \mathbb{R} \to \mathbb{R}$ on the closed interval $[l_j, u_j]$ as follows:

$$g_j(x_j) = c_j x_j + d_j,$$

where

$$\begin{aligned} c_j &= [f_j(u_j) - f_j(l_j)]/(u_j - l_j), \\ d_j &= f_j(l_j) - \{[f_j(u_j) - f_j(l_j)]/(u_j - l_j)\} l_j \\ &= f_j(l_j) - c_j l_j. \end{aligned}$$

Thus,

$$\begin{aligned} g_j(x_j) &= c_j x_j + d_j \\ &= c_j x_j + f_j(l_j) - c_j l_j. \end{aligned}$$

Then, the objective function

$$f(x) = \sum_{j=1}^{n} f_j(x_j)$$

is approximated by the linear function

$$\begin{aligned} g(x) &= \sum_{j=1}^{n} g_j(x_j) \\ &= \sum_{j=1}^{n} [c_j x_j + f_j(l_j) - c_j l_j] \\ &= c^T x + [f(l) - c^T l] \\ &= c^T x + \delta, \end{aligned}$$

on the set $\mathcal{P} = \mathcal{A} \cap \mathcal{T}$, where

$$\delta = f(l) - c^T l.$$

From the properties of the function f, it is easy to show that the following relation:

$$f(x) \geq g(x) = c^T x + \delta$$

holds for all $x \in \mathcal{P}$.

2.2. Computing the Lower Bound for the Objective Value of (P). The lower bound for the objective value of (P) can be computed using the following linear programming problem:

$$(\mathrm{P_{LP}}) \quad \min_{x\in\mathcal{P}} c^T x+\delta.$$

Let us denote the optimal solution of ($\mathrm{P_{LP}}$) by $\tilde{x}$. Then,

$$\beta = g(\tilde{x}) = c^T\tilde{x}+\delta,$$

which is the optimal objective value of ($\mathrm{P_{LP}}$). Therefore, the relation

$$\begin{aligned}\beta &= c^T\tilde{x}+\delta\\ &\le f(x)\\ &\le f(\tilde{x})+[\nabla f(\tilde{x})]^T(x-\tilde{x})\end{aligned}$$

holds for all $x\in\mathcal{P}$; namely, a lower bound is obtained for the optimal objective value problem (P). The second inequality holds because of the concavity of the function f, since the linear function[4]

$$\bar{f}(x) := f(\tilde{x})+[\nabla f(\tilde{x})]^T(x-\tilde{x}) \tag{1}$$

is the tangent of f in $\tilde{x}\in\mathcal{P}$.

2.3. Optimality Criterion of the Relaxed LP Problem. Without loss of generality, we may be consider the relaxed LP problem of (P) in the following form:

$$\begin{aligned}(\mathrm{P_{LP}}) \quad &\min \; c^T x,\\ &\text{s.t} \quad Ax\le b,\\ &\qquad\; l\le x\le u.\end{aligned}$$

in which the constant δ has been deleted from the objective function. Let us denote the optimal solutions of problem ($\mathrm{P_{LP}}$) by

$$\mathcal{P}_c^* = \{x^*\in\mathcal{P} : c^T x^*\le c^T x,\, x\in\mathcal{P}\}.$$

The sensitivity analysis requires additional notation related to linear programming.

[4]The differentiability condition can be disregarded, since the concave function f has a subgradient at every inner point of its domain (Refs. 25–26) and so a subgradient can be considered instead of $\nabla f(\tilde{x})$.

Let us denote the index set of the optimal basis by $\mathcal{J}_B \subset \mathcal{J}$, while $\mathcal{J}_N$ contains the indices of nonbasic variables. The vectors $\{a_j : j \in \mathcal{J}_B\}$ are linearly independent and obviously

$$\mathcal{J} = \mathcal{J}_B \cup \mathcal{J}_N \quad \text{and} \quad \mathcal{J}_B \cap \mathcal{J}_N = \emptyset.$$

Furthermore, the index set $\mathcal{J}_N^l \subset \mathcal{J}_N$ $[\mathcal{J}_N^u \subset \mathcal{J}_N]$ contains those indices which are at their lower [upper] bound in this basis. In our problem (P), the index set of nonbasic variables $\mathcal{J}_N$ is partitioned into two subsets as follows:

$$\mathcal{J}_N = \mathcal{J}_N^l \cup \mathcal{J}_N^u.$$

As well, we let

$$\bar{A} = [B^{-1}A]$$

and c_B denotes the vector which contains the objective coefficients of the basic variables.

For any $\bar{x} \in \mathcal{P}$ basic feasible solution of problem (P_{LP}), the following statements are true:

$$\begin{aligned} &l_i \le \bar{x}_i \le u_i, && \text{for all } i \in \mathcal{J}_B, \\ &\bar{x}_i = l_i, && \text{for all } i \in \mathcal{J}_N^l, \\ &\bar{x}_i = u_i, && \text{for all } i \in \mathcal{J}_N^u. \end{aligned}$$

As a consequence, if a basic partition of the index set $\mathcal{J}$ is given as $(\mathcal{J}_B, \mathcal{J}_N^l, \mathcal{J}_N^u)$, then the basic variables vector can be computed as follows:

$$\bar{x}_B = B^{-1}b - \sum_{j \in \mathcal{J}_N^l} l_j \bar{a}_j - \sum_{j \in \mathcal{J}_N^u} u_j \bar{a}_j,$$

where $\bar{a}_j$ denotes the jth column vector of the matrix $\bar{A}$.

The dual problem (P_{LP}) has the form

$$\begin{aligned} (D_{LP}) \quad &\max && -b^T y + l^T z - u^T s, \\ &\text{s.t.} && -A^T y + z - s = c, \\ & && y \ge 0, \quad z \ge 0, \quad s \ge 0, \end{aligned}$$

and

$$\mathcal{D} = \{(y, z, s) : -A^T y + z - s = c, y \ge 0, z \ge 0, s \ge 0\}$$

denotes the set of dual feasible solutions. Let us consider the weak duality theorem related to problems (P_{LP}) and (D_{LP}).

Proposition 2.3. Let $x \in \mathcal{P}$ and $(y, z, s) \in \mathcal{D}$ be vectors. Then, the inequality

$$c^T x \geq -b^T y + l^T z - u^T s \tag{2}$$

holds. In (2), equality holds if and only if

$$\begin{aligned} 0 &= c^T x + b^T y - l^T z + u^T s \\ &= y^T (b - Ax) + z^T (x - l) + s^T (u - x). \end{aligned}$$

Now, we are ready to introduce the necessary and sufficient optimality criteria of problems (P_{LP}) and (D_{LP}) as

$$\begin{aligned} &Ax \leq b, \quad l \leq x \leq u, \\ &-A^T y + z - s = c, \quad y \geq 0, \quad z \geq 0, \quad s \geq 0 \\ &y(b - Ax) = 0, \quad z(x - l) = 0, \quad s(u - x) = 0, \end{aligned}$$

where $y(b - Ax)$, $z(x - l)$, $s(u - x)$ denote the Hadamard (coordinatewise) product of the corresponding vectors.

Assuming that $x^* \in \mathcal{P}_c^*$ is a basic solution belonging to the basis B and that

$$y^* = c_B^T B^{-1} \geq 0,$$

we get that:

(i) if $j \in \mathcal{J}_B$, $l_j < x_j^* < u_j$, $z_j = 0$, $s_j = 0$, then

$$-a_j^T y = c_j;$$

(ii) if $j \in \mathcal{J}_N^l$, $l_j = x_j^*$, $z_j \geq 0$, $s_j = 0$, then

$$z_j = c_j + a_j^T y \geq 0;$$

(iii) if $j \in \mathcal{J}_N^u$, $u_j = x_j^*$, $z_j = 0$, $s_j \geq 0$, then

$$-s_j = c_j + a_j^T y \leq 0.$$

Finally, we obtain a basic solution $x^* \in \mathcal{P}$, which is optimal if and only if

$$y^* = c_B^T B^{-1} \geq 0, \tag{3}$$

$$-c_B^T B^{-1} a_j \leq c_j, \quad \text{for any } j \in \mathcal{J}_N^l, \tag{4}$$

$$-c_B^T B^{-1} a_j \geq c_j, \quad \text{for any } j \in \mathcal{J}_N^u. \tag{5}$$

3. Sufficient Optimality Criterion

Here, we formulate and prove a sufficient optimality criterion for problem (P) with regard to an extremal point of the set $\mathcal{P}$ concerning a basic solution.

Let us define the set $\mathcal{H} \subseteq \mathbb{R}^n$, which contains the coefficients of the corresponding relaxed linear, objective functions. The set $\mathcal{H}$ must be such that, if the optimal solutions of the linear programming problem related to the elements of set $\mathcal{H}$ were known, then the optimal solution of (P) could be generated as well. Otherwise, it would be impossible to identify the optimal solution of problem (P) using optimization methods based on linear relaxation.

At first, we examine $\mathcal{H}$ in general terms and utilize the most important properties of it. Later, we define sets with simple structure that approximates the set $\mathcal{H}$.

The next lemma states the existence of a vector $h \in \mathbb{R}^n$ for an optimal basic of solution $\hat{x}$ of problem (P) such that $\hat{x} \in P_h^*$ holds. Namely, $\hat{x}$ is an element of the set of optimal solutions of the relaxed linear programming problem with objective function coefficient h.

Lemma 3.1. Consider problem (P). The optimal solution of (P) is denoted by $\hat{x}$ and $f(\hat{x}) = \min_{x \in \mathcal{P}} f(x)$. Then,

$$\bar{f}(\hat{x}) = \min_{x \in \mathcal{P}} \bar{f}(x),$$

where

$$\bar{f}(x) = [\nabla f(\hat{x})]^T (x - \hat{x}) + f(\hat{x})$$

is a linear function defined by equation (1).

Proof. The following inequality holds because of the concavity of function f,

$$f(x) \leq \bar{f}(x) = [\nabla f(\hat{x})]^T (x - \hat{x}) + f(\hat{x}),$$

with strict equality at $\hat{x}$; namely,

$$f(\hat{x}) = \bar{f}(\hat{x}).$$

Consider the linear programming problem with objective function $\bar{f}(x)$. Then,

$$\begin{aligned} f(\hat{x}) &= \min_{x\in\mathcal{P}} f(x) \\ &\le \min_{x\in\mathcal{P}} \bar{f}(x) \\ &\le \bar{f}(\hat{x}) \\ &= f(\hat{x}), \end{aligned}$$

from which

$$\min_{x\in\mathcal{P}} \bar{f}(x) = \bar{f}(\hat{x})$$

is obtained. □

In the above lemma, the differentiability of function f_j' on the interval $[l_j, u_j]$ is used. It is easy to show that, if f is not differentiable at $\bar{x}$, then any inner point of the set of subgradients is also suitable for the function $\bar{f}$.

It has been proved that there exists a vector h such that the optimal solution $\hat{x}$ of the relaxed linear programming problem is also an optimal solution for problem (P). Thus, the set $\mathcal{H}$ should contain the vectors $\nabla f(x), x \in \mathcal{P}$.

For any $\bar{x} \in \mathcal{P}$ basic solution, the set $\mathcal{C}_B$ can be formulated as follows:

$$\mathcal{C}_B = \{c \in \mathbb{R}^n: \text{ vector } c \text{ satisfies equations (3)--(5)}\}.$$

The set $\mathcal{C}_B$ contains such vectors c for which $\bar{x}$ is an optimal solution of the linear programming problem

$$(\mathrm{P}_c) \qquad \min_{x\in\mathcal{P}} \; c^T x.$$

Obviously, the set $\mathcal{C}_B$ is not empty.

The proof of the following statement about any linear programming relaxation of problem (P) is quite easy.

Proposition 3.1. Consider the basic solution $\bar{x} \in \mathcal{P}$ with basis B and let $\bar{h} \in \mathcal{H}$ be a given vector. If $\bar{h} \in \mathcal{C}_B$, then $\bar{x}$ is an optimal solution of the following linear programming problem:

$$(\mathrm{P}_{\bar{h}}) \qquad \min_{x\in\mathcal{P}} \; \bar{h}^T x;$$

namely $\bar{x} \in \mathcal{P}^*_{\bar{h}}$, where $\mathcal{P}^*_{\bar{h}}$ denotes the set of optimal solutions of problem $(\mathrm{P}_{\bar{h}})$.

From this result, it follows that

$$\text{if} \quad \mathcal{H} \subseteq \mathcal{C}_B, \quad \text{then } \bar{x} \in \mathcal{P}^*_{\bar{h}}, \quad \text{for any } h \in \mathcal{H}. \tag{6}$$

We are now ready to introduce and prove our main result, the sufficient optimality criterion for the linearly-constrained separable concave minimization problem (P).

Theorem 3.1. Consider the linearly-constrained separable concave minimization problem (P) and suppose that the functions f_j are strictly concave. Let $\bar{x} \in \mathcal{P}$ be a basic solution with basis B such that $\mathcal{H} \subseteq \mathcal{C}_B$ holds; then, $\mathcal{P}^* = \{\bar{x}\}$.

Proof. Since $\mathcal{H} \subseteq \mathcal{C}_B$, thus $\bar{x} \in \mathcal{P}^*_h$ holds for any $h \in \mathcal{H}$. As it was mentioned, the compactness of $\mathcal{P}$ implies that the global minimum is obtained. Further, because the function f is concave, the global minimum is found at an extremal point $\hat{x} \in \mathcal{P}$ as well.

Let $\hat{h} = \nabla f(\hat{x})$. Since Lemma 3.1 asserts that $\hat{x} \in \mathcal{P}^*_{\hat{h}}$, otherwise $\bar{x} \in \mathcal{P}^*_{\hat{h}}$, the following relations hold;

$$f(\hat{x}) = \bar{f}(\hat{x}) = \bar{f}(\bar{x}) > f(\bar{x}), \tag{7}$$

which constitute a contradiction. Thus, $\hat{x} = \bar{x}$ and it follows that $\mathcal{P}^* = \{\bar{x}\}$. □

The strict inequality comes from the strict concavity. If the condition of strict concavity is removed from Theorem 3.1, then the inequality (7) will be modified,

$$f(\bar{x}) \geq f(\hat{x}) = \bar{f}(\hat{x}) = \bar{f}(\bar{x}) \geq f(\bar{x}),$$

so $f(\bar{x}) = f(\hat{x})$, and thus $\bar{x} \in \mathcal{P}^*$, but the equality $|\mathcal{P}^*| = 1$ cannot be guaranteed.

It has been proved that the sufficient optimality criterion for a basic solution $\bar{x} \in \mathcal{P}$ of problem (P) with basis B is that

$$\mathcal{H} \subseteq \mathcal{C}_B.$$

4. On the Set $\mathcal{H}$

The set $\mathcal{H}$ determines the strength of the optimality criteria. For some problem (P) and given basis, the set $\mathcal{H}$ can be computed easily; however, in general, the set $\mathcal{H}$ is nonlinear and nonconvex. Obviously, the best approximation of the set $\mathcal{H}$ is our aim.

The set $\mathcal{H}$ contains vectors that are the linear relaxation of the function f at some feasible points of the problem. These vectors are related closely to the derivative of the function f. Checking optimality is equivalent to investigating the relation between the two sets $\mathcal{C}_B$ and $\mathcal{H}$. Determining the set $\mathcal{H}$ involves taking into account that the relation between these two sets is to be examined easily. One way to determine $\mathcal{H}$ is to consider the range of the derivative of the function f on the set $\mathcal{P}$. If f is strictly concave then, f' is strictly decreasing, so it has an inverse function g. Thus, the set

$$\mathcal{F} = \{y : Ag(y) = b \text{ and } l \leq g(y) \leq u\}$$

is the range of f' on $\mathcal{P}$, which is suitable for the set $\mathcal{H}$. However, the set $\mathcal{F}$ can have a complicated structure (i.e. nonlinear, nonconvex); so, if we choose $\mathcal{H} = \mathcal{F}$, then deciding whether $\mathcal{H} \subseteq \mathcal{C}_B$ is as difficult as to solve the original problem (P).

If the function f is quadratic, then the function g is linear, so $\mathcal{H} = \mathcal{F}$ is also a polyhedron. In the general case, we approximate the set $\mathcal{F}$, so that it is contained in a set which has a less complicated structure (i.e., polyhedron).

Obviously, the determination of $\mathcal{H}$ is influenced greatly by the property of the function f (strict concavity, differentiability, etc.). On the other hand, if the structure of the set $\mathcal{H}$ is complicated (not a polyhedron), then verifying the optimality criterion (6) can be very difficult. For this reason, it is worth determining a set having a simple structure (hyperrectangle) that encloses the set $\mathcal{H}$. If the set approximating $\mathcal{H}$ is based on the properties of the objective function f and on the bounds l and u, we can get

$$\mathcal{H}_f = \{h \in \mathbb{R}^n : h_j \in [f'_{j-}(u_j), f'_{j+}(l_j)]\}$$

and $\mathcal{H} \subseteq \mathcal{H}_f$ holds. Further, let us determine a set approximating $\mathcal{H}$ at a given basic solution $\bar{x} \in \mathcal{P}$ by using the hyperrectangle

$$\mathcal{H}_{f,\bar{x}} = \{h \in \mathbb{R}^n : h_j \in [c^l_j, c^u_j]\}.$$

This set will contain the coefficients of all possible relaxed linear functions, where

$$c_j^u = \begin{cases} m(l_j, \bar{x}_j), & \bar{x}_j \neq l_j, \\ f'_{j+}(l_j), & \text{otherwise}, \end{cases}$$

$$c_j^l = \begin{cases} m(\bar{x}_j, u_j), & \bar{x}_j \neq u_j, \\ f'_{j-}(u_j), & \text{otherwise}. \end{cases}$$

Philips and Rosen (Ref. 1) used also the set $\mathcal{H}_{f,\bar{x}}$. From Propositions 2.1 and 2.2, we derive the inequalities

$$f'_{j-}(u_j) \leq c_j^l = m(\bar{x}_j, u_j) \leq m(l_j, \bar{x}_j) = c_j^u \leq f'_{j+}(l_j). \tag{8}$$

Therefore, the relationship $\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{H}_f$ holds.[5]

Philips and Rosen (Ref. 1) have presented the following result: Take all linear programming problems with the same feasible solution set $\mathcal{P}$ such that the objective function coefficients of the problems are vertices of $\mathcal{H}_{f,\bar{x}}$. Solve these $2^{\dim(H_{f,\bar{x}})}$ linear programming problems and check if all have a common optimal solution. If a common optimal solution exists for all these linear programming relaxations of problem (P), then that solution will be a global optimal solution of problem (P) as well. We modify slightly the Philips and Rosen (Ref. 1) question; namely, we are interested in determining whether the relation

$$\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{C}_B \tag{9}$$

is true. Obviously, it is enough to decide whether the extremal points of the hyperrectangle $\mathcal{H}_{f,\bar{x}}$ are elements of $\mathcal{C}_B$. This observation can save a significant amount of computation, but is still needs verification of whether or not the vertices [exponential number of points $2^{\dim(H_{f,\bar{x}})}$] belong to the set $\mathcal{C}_B$.

We are going to show that checking the inclusion (9) can be done much more efficiently; namely, it is enough to select n special vertices of $H_{f,\bar{x}}$ to test if they satisfy the inequalities that define C_B. If all these vertices of the set $H_{f,\bar{x}}$ belong to C_B, then (9) holds. The function (9) is a practically applicable sufficient optimality criterion for problem (P). If (9) holds, then $\bar{x} \in \mathcal{P}$ is a global optimal solution of problem (P).

[5] If the functions f_j are strictly concave, then the inequalities in (8) are fulfilled strictly.

4.1. Defining a Test Point. Instead of checking the inclusion $\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{C}_B$, we define a test point chosen from the set $\mathcal{H}_{f,\bar{x}}$ for each inequality of the system defined in (3)–(5). Now, the coefficients of the linear objective function are the unknown variables in the inequality system (3)–(5). If there is no test point that violates at least one inequality, then the inclusion, $\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{C}_B$ should hold.

Let us define a test point that belongs to $\mathcal{H}_{f,\bar{x}}$ and violates the constraint indexed by $j \in \mathcal{J}_N^l$,

$$-c_B^T B^{-1} a_j = -c_B^T \bar{a}_j \le c_j.$$

This means that we choose a vertex of $\mathcal{H}_{f,\bar{x}}$ that increases the left side of the inequality and decreases the right side as much as possible. Therefore, the test point $\bar{h}_j$ can be defined as follows:

$$\bar{h}_{ij} = \begin{cases} c_j^l, & \text{if } i = j, \\ c_j^l, & \text{if } \bar{a}_{ij} > 0, \ i \in \mathcal{J}_B, \\ c_j^u, & \text{if } \bar{a}_{ij} < 0, \ i \in \mathcal{J}_B, \\ h_{ij}, & \text{if } i \notin (\mathcal{J}_B \setminus \{i : \bar{a}_{ij} = 0\}) \cup \{j\}, \text{where } h_{ij} \in \left[c_i^l, c_i^u\right]. \end{cases}$$

It is obvious that $\bar{h}_j \in \mathcal{H}_{f,\bar{x}}$ holds. From the construction of the test point, it is clear that the inequality

$$\bar{h}_B^T \bar{a}_j + \bar{h}_{jj} \le h_B^T \bar{a}_j + h_{jj}$$

holds for any $h \in \mathcal{H}_{f,\bar{x}}$, which is

$$-\bar{h}_B^T \bar{a}_j - \bar{h}_{jj} \ge -h_B^T \bar{a}_j - h_{jj}. \tag{10}$$

Now, if the test point does not violate the inequality, that is, if

$$0 \ge -\bar{h}_B^T \bar{a}_j - \bar{h}_{jj}, \tag{11}$$

then based on (10) and (11), there is no element of the set $\mathcal{H}_{f,\bar{x}}$ that can violate the inequality $j \in \mathcal{J}_N^l$. In general, the test point $\bar{h}_k$ for any index $k \in \mathcal{J}_N^l \cup \mathcal{J}_N^u$ will be defined using the sets $\mathcal{J}_i^+, \mathcal{J}_i^-$ and $i \in \mathcal{J}_B$ as follows:

$$\bar{h}_{ik} = \begin{cases} c_i^l, & \text{if } k \in \mathcal{J}_i^-, i \in \mathcal{J}_B, \\ c_i^u, & \text{if } k \in \mathcal{J}_i^+, i \in \mathcal{J}_B, \\ c_k^l, & \text{if } i = k \text{ and } k \in \mathcal{J}_N^l, \\ c_k^u, & \text{if } i = k \text{ and } k \in \mathcal{J}_N^u, \\ h_i, & i \notin (\mathcal{J}_B \setminus \{i : \bar{a}_{ik} = 0\}) \cup \{k\}, \text{where } h_i \in \left[c_i^l, c_i^u\right], \end{cases}$$

with

$$\mathcal{J}_i^+ = \left\{k \in \mathcal{J}_N^l : \bar{a}_{ik} < 0\right\} \cup \left\{k \in \mathcal{J}_N^u : \bar{a}_{ik} > 0\right\},$$
$$\mathcal{J}_i^- = \left\{k \in \mathcal{J}_N^l : \bar{a}_{ik} > 0\right\} \cup \left\{k \in \mathcal{J}_N^u : \bar{a}_{ik} < 0\right\}.$$

Based on these observations, we obtain the following proposition.

Proposition 4.1. If the test point $\bar{h}_k$ does not violate the inequality $k \in \mathcal{J}_N^l \cup \mathcal{J}_N^u$, then no point $h \in \mathcal{H}_{f,\bar{x}}$ violates either.

Moreover, in the case of $j \in \mathcal{J}_N^l [j \in \mathcal{J}_N^u]$,

$$-\bar{h}_{B,j}^T \bar{a}_j > c_j^l \ \ [-\bar{h}_{B,j}^T \bar{a}_j < c_j^u],$$

the test point $\bar{h}_j$ violates the optimality criterion which belongs to the variable j. We can determine a test point for testing the inequality system

$$\bar{h}_B^T B^{-1} \geq 0.$$

Let the matrix $\bar{B} = B^{-1}$ and let $\bar{b}_i$ denote the ith column of the matrix $\bar{B}$. Then,

$$\bar{h}_{ji} = \begin{cases} c_i^l, & \text{if } b_{ji} > 0,\ j \in \mathcal{J}_B, \\ c_i^u, & \text{if } b_{ji} < 0,\ j \in \mathcal{J}_B, \\ h_i, & \text{if } j \in \mathcal{J}_N^l \cup \mathcal{J}_N^u \cup \{j \in \mathcal{J}_B : \bar{b}_{ji} = 0\}, \text{where } h_i \in \left[c_j^l, c_j^u\right]. \end{cases}$$

In this case, $\bar{h}_{j,B}^T \bar{b}_i \geq 0$ holds, and any vector $h \in \mathcal{H}_{f,\bar{x}}$ satisfies the ith nonnegativity condition. Therefore, instead of testing 2^n vertices of the hyperrectangle $\mathcal{H}_{f,\bar{x}}$, it is enough to determine n test points in order to check whether or not the inclusion $\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{C}_B$ holds.

Let us introduce the index set $\mathcal{K}$,

$$\mathcal{K} = \{i : \bar{h}_i \text{ test point violates the } i\text{th inequality}\}.$$

It is obvious that the equality $\mathcal{K} = \emptyset$ leads to

$$\mathcal{H}_{f,\bar{x}} \subseteq \mathcal{C}_B;$$

thus, $\bar{x} \in \mathcal{P}^*$ holds. The determination of whether the basic solution $\bar{x} \in \mathcal{P}$ is an optimal solution for problem (P) can be performed as follows:

Step 1. Generate the set $\mathcal{H}_{f,\bar{x}}$.
Step 2. Using the matrices B^{-1} and $B^{-1}A_N$, generate the test point $\bar{h}_j$.

Step 3. Perform the appropriate verification of the test points. If there is no index j for which the test point $\bar{h}_j$ violates the jth condition, then $\bar{x}$ is an optimal solution for problem (P).

Nevertheless, if any test point $\bar{h}_j$ can be determined that violates the jth condition, can we conclude that $\bar{x} \in \mathcal{P}$ is not an optimal solution of the problem (P)? Unfortunately, we cannot, because we do not know how good the approximation of the set $\mathcal{H}$ is by the set $\mathcal{H}_{f,\bar{x}}$.

Since the sets $\mathcal{H}$ and $\mathcal{H}_{f,\bar{x}}$ depend significantly on the bounds l_j and u_j, it is expected that, if the diameter of the set $\mathcal{H}_{f,\bar{x}}$ is rapidly decreased, then a branch-and-bound type algorithm is efficient for solving problem (P). In the case of branch-and-bound type algorithms, the partition is to be performed in a way that the diameter of the set $\mathcal{H}_{f,\bar{x}}$ decreases. The numerical testing of the branch-and-bound algorithm, using the test point introduced above, is in progress.

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